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Remarks on strict efficiency in scalar and vector optimization

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Abstract The aim of this paper is to study optimality conditions for strict local minima to constrained mathematical problems governed by scalar and vectorial mappings. Unlike other papers in literature dealing with strict efficiency, we work here with mappings defined on infinite dimensional normed vector spaces. Firstly, we (mainly) consider the case of non-smooth scalar mappings and we use the Fréchet and Mordukhovich subdifferentials in order to provide optimality conditions. Secondly, we present some methods to reduce the study of strict vectorial minima to the case of strict scalar minima by means of some scalarization techniques. In this vectorial framework we treat separately the case where the ordering cone has non-empty interior and the case where it has empty interior.

Keywords Scalar optimization · Vector optimization · Strict efficiency · Basic subdifferential · Fréchet subdifferential

Mathematics Subject Classification (2000) 90C29 · 90C26 · 49J52

1 Introduction

The study of classical local solutions is well represented in the literature dealing with optimization problems. However, in the last decades another types of solutions were considered and, among these concepts, the strict solutions (cf. Definition 3.2 below) are of particular importance for several reasons including that one that strict solutions are, in contrast to regular minimum points, the most likely to be found by numerical algorithms. Even if the name of this concept varies in different papers, nowadays there exists an important literature dedicated to this topic. We quote here, far from being exhaustive, [5,9,14] and the reference therein for historical comments and different approaches in both scalar and vectorial cases.

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In all quoted papers the framework is restricted to the case of finite dimensional normed vector spaces and the optimality conditions are expressed either in terms of various tangent cones and related directional derivatives or in terms of normal cones and (convex) subdifferential. The aim of this paper is to reconsider the concept of strict solution in infinite dimensional setting and to obtain separate necessary and sufficient optimality conditions in terms of Mordukhovich and Fréchet subdifferentials and the corresponding normal cones. Our approach is based, in the scalar case, on several calculus rules for these subdifferentials, while, in the vectorial case, we use some well-known scalarization functionals to reduce the study to the scalar case. The approach we employ here has not been taken previously (up to our knowledge) and it allows us to obtain several new results in what concerns the optimality conditions for strict solutions. We point out as well the differences and the similarities between scalar and vectorial cases and, when we deal with necessary optimality conditions, we recover some known conditions for classical concepts of solution.

The paper is organized as follows. In the second section we present the basic notations and results we work with in the rest of the paper. In the third section we consider the notion of strict solution for an usual scalar problem with geometric constraints and we derive separate necessary and sufficient optimality conditions for this type of solution. We firstly discuss some basic sufficient optimality conditions that one can meet in literature for the case of differentiable or convex mappings. Then, we formulate similar conditions by means of generalized differentiation objects in the case of nonsmooth functions defined on infinite dimensional normed vector spaces. The last section is devoted to strict solutions for optimization problems with vectorial cost functions. In this case we divide the discussion into two main cases, namely the case where the ordering cone is solid (i.e., it has nonempty topological interior) and the opposite case where the ordering cone is non-solid. In the first case we reinterpret the definition of vectorial strict minima by means of a well-known scalarization functional having good calculus properties. This allows us to present optimality conditions in terms of basic subdifferential and then to cover the scalar case in many aspects. The particular case of convex vectorial mappings is briefly considered as well. In the second situation (of non-solid cones) which is theoretically more difficult, we present a modification of the definition of strict efficiency which proves to be suitable in order to formulate meaningful optimality conditions.

2 Preliminaries

Let X a normed vector space. We denote by $D(x, \varepsilon)$ and $B(x, \varepsilon)$ the closed and the open ball with center x and radius $\varepsilon > 0$. Sometimes, for simplicity, the closed unit ball of X and the unit sphere are denoted as U_X and S_X, respectively. For a nonempty set $A \subset X$ we denote by $\operatorname{int} A$, $\operatorname{cl} A$ the topological interior and the topological closure, respectively. The notation X^* designates the topological dual of X. The symbol $\mathbb{R}_+$ is for the set of nonnegative real numbers.

Let $S \subset X$ be a nonempty set and $\overline{x} \in \operatorname{cl} S$. The Bouligand (contingent) cone to S at $\overline{x}$ is defined as

$$T_B(S, \overline{x}) = \{u \in X \mid \exists t_n \downarrow 0, \exists u_n \to u, \forall n \in \mathbb{N}, \overline{x} + t_n u_n \in S\}$$

where $(t_n) \downarrow 0$ means $(t_n) \subset (0, \infty)$ and $t_n \to 0$. This cone is always closed but in general it is not convex. It is well known that if S is convex, then the contingent cone coincides with the tangent cone in the sense of convex analysis given by:

$$T(S, \overline{x}) = \text{cl}\,(\mathbb{R}_+(S - \overline{x})).$$

In general, we define the contingent normal cone to S at $\overline{x}$ as

$$N_B(S, \overline{x}) := T_B(S, \overline{x})^- := \left\{u^* \in X^* \mid u^*(u) \leq 0, \forall u \in T_B(S, \overline{x})\right\}.$$

The following notion was introduced in [11, Definition 4.1].

Definition 2.1 Let $S \subset X$ be a nonempty set, $\overline{x} \in S$ and $K \subset X$ be a closed cone. One says that S is approximated at $\overline{x}$ by K if there exists a function $\alpha : S \to K$ s.t.

$$\lim_{x \xrightarrow{S} \overline{x}} \frac{\alpha(x) - (x - \overline{x})}{\|x - \overline{x}\|} = 0, \tag{1}$$

where $x \xrightarrow{S} \overline{x}$ means $x \to \overline{x}$ and $x \in S$.

Concerning this notion some useful observations are listed below:

- If S is approximated at $\overline{x}$ by K then $T_B(S, \overline{x}) \subset K$. Indeed, let $\alpha : S \to K$ satisfying (1) and take $u \in T_B(S, \overline{x}) \setminus \{0\}$. Then there exist $t_n \downarrow 0$, $u_n \to u$ s.t. $\overline{x} + t_n u_n \in S$ for every $n \in \mathbb{N}$. By (1) we get $u = \lim t_n^{-1} \alpha\,(\overline{x} + t_n u_n) \in K$. Therefore, $T_B(S, \overline{x}) \subset K$.
- If X is finite dimensional, then S is approximated at every $\overline{x} \in S$ by $T_B(S, \overline{x})$ (see [11, Theorem 4.2 (ii)]);
- If S is convex, then it is approximated at every $\overline{x} \in S$ by $T_B(S, \overline{x})$: simply take $\alpha(x) = x - \overline{x}$ for every $x \in S$.

In this paper we use as well the generalized differentiation objects in the sense of Mordukhovich. The definitions below are from the comprehensive monograph [12] and we present it here for the sake of completeness.

Definition 2.2 Let X be a Banach space and $S \subset X$ be a nonempty subset of X and let $x \in S$.

(i) The basic (or limiting, or Mordukhovich) normal cone to S at x is

$$N(S, x) := \{x^* \in X^* \mid \exists \varepsilon_n \downarrow 0, x_n \xrightarrow{S} x, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}_{\varepsilon_n}(S, x_n), \forall n \in \mathbb{N}\}$$

where $\widehat{N}_\varepsilon(S, z)$ denotes the Fréchet set of ε-normals ($\varepsilon \geq 0$) to S at a point $z \in S$, given as

$$\widehat{N}_\varepsilon(S, z) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} z} \frac{x^*(u - z)}{\|u - z\|} \leq \varepsilon \right\}.$$

(ii) Let $f : X \to \overline{\mathbb{R}}$ be finite at $\overline{x} \in X$; the Fréchet subdifferential of f at $\overline{x}$ is the set

$$\hat{\partial} f(\overline{x}) := \left\{x^* \in X^* \mid (x^*, -1) \in \widehat{N}\,(\text{epi}\, f, (\overline{x}, f(\overline{x})))\right\}$$

and the basic (or limiting, or Mordukhovich) subdifferential of f at $\overline{x}$ is

$$\partial f(\overline{x}) := \left\{x^* \in X^* \mid (x^*, -1) \in N\,(\text{epi}\, f, (\overline{x}, f(\overline{x})))\right\}$$

where epi f denotes the epigraph of f.

For $\varepsilon = 0$ the set $\widehat{N}_\varepsilon(S, z)$ is a convex cone which is denoted by $\widehat{N}(S, z)$. If S is convex and $z \in S$, both cones $\widehat{N}(S, z)$ and $N(S, z)$ do coincide with $N_B(S, z)$ and have the following particular form:

$$N(S, \overline{x}) = \widehat{N}(S, z) = N_B(S, z) = \left\{u^* \in X^* \mid u^*(x - z) \leq 0, \forall x \in S\right\}.$$

If f is convex then both $\hat{\partial} f(\overline{x})$, $\partial f(\overline{x})$ do coincide with the classical Fenchel subdifferential.

If X is an Asplund space (i.e., a Banach space where every convex continuous function is generically Fréchet differentiable) then one has a simpler form for the basic normal cone, that is:

$$N(S, x) := \{x^* \in X^* \mid \exists x_n \overset{S}{\to} x, x_n^* \overset{w^*}{\to} x^*, x_n^* \in \widehat{N}(S, x_n), \forall n \in \mathbb{N}\}$$

and, accordingly,

$$\partial f(\overline{x}) = \limsup_{x \overset{f}{\to} \bar{x}} \hat{\partial} f(x),$$

where $x \overset{f}{\to} \overline{x}$ means $x \to \overline{x}$ and $f(x) \to f(\overline{x})$. In particular, $\hat{\partial} f(\overline{x}) \subset \partial f(\overline{x})$ and if $\overline{x}$ is a local minimum point of f then $0 \in \hat{\partial} f(\overline{x})$. If δ_Ω denotes the indicator function associated with a nonempty set $\Omega \subset X$ (i.e., $\delta_\Omega(x) = 0$ if $x \in \Omega$, $\delta_\Omega(x) = \infty$ if $x \notin \Omega$), then for any $\overline{x} \in \Omega$, $\partial \delta_\Omega(\overline{x}) = N(\Omega, \overline{x})$. In contrast to the Fréchet subdifferential, the basic subdifferential satisfies a robust calculus rule: if X is an Asplund space, $f_1, f_2, \ldots, f_{n-1}$ are Lipschitz around $\overline{x}$ and f_n is lower semicontinuous around this point, then

$$\partial \left(\sum_{i=1}^{n} f_i \right)(\overline{x}) \subset \sum_{i=1}^{n} \partial f_i(\overline{x}). \tag{2}$$

We also remind (see, e.g., [12, Theorem 1.88]) the following characterization of the elements of the Fréchet subdifferential. Let $f : X \to \overline{\mathbb{R}}$ be a function finite at $\overline{x}$.

- If $x^* \in X^*$ and there exists a real-valued function s defined on a neighborhood of $\overline{x}$ s.t. s is differentiable at $\overline{x}$, $\nabla s(\overline{x}) = x^*$ and $f - s$ achieves a local minimum at $\overline{x}$, then $x^* \in \hat{\partial} f(\overline{x})$.
- For every $x^* \in \hat{\partial} f(\overline{x})$ there exists a function $s : X \to \mathbb{R}$ with $s(\overline{x}) = f(\overline{x})$ and $s(x) \leq f(x)$ for every $x \in X$ s.t. s is Fréchet differentiable at $\overline{x}$ and $\nabla s(\overline{x}) = x^*$.

The set S is called normally regular at $\overline{x} \in S$ if $N(S, \overline{x}) = \widehat{N}(S, \overline{x})$. A function f is called lower regular at $\overline{x} \in \operatorname{dom} f$ if $\hat{\partial} f(\overline{x}) = \partial f(\overline{x})$. Of course, every convex set S is normally regular at $\overline{x}$ whenever $\overline{x} \in S$ and every convex function is lower regular at $\overline{x}$ whenever $\overline{x} \in \operatorname{dom} f$.

We end this section with an interesting exact calculus rule for the Fréchet subdifferential (see [13]): if X is a Banach space, f_1, f_2 are arbitrary extended-real-valued function finite at $\overline{x}$ and

$$\hat{\partial}^+ f_2(\overline{x}) := -\hat{\partial}(-f_2)(\overline{x})$$

is nonempty then

$$\hat{\partial}(f_1 + f_2)(\overline{x}) \subset \bigcap_{x^* \in \hat{\partial}^+ f_2(\overline{x})} [x^* + \hat{\partial} f_1(\overline{x})]. \tag{3}$$

3 Scalar case

Let X be a normed vector space, $f : X \to \mathbb{R}$ be a function and $S \subset X$ be a closed set. In this section we consider the scalar problem with geometrical constraint associated to f and S:

$$(P_S) \qquad \min f(x), \quad x \in S.$$

As usual, $\overline{x} \in S$ is called local solution of (P_S) if there exists a neighborhood U of $\overline{x}$ s.t.

$$f(\overline{x}) \le f(x), \quad \forall x \in U \cap S.$$

In order to motivate the subsequent ideas, we remind that in the case where f is differentiable at $\overline{x}$ one has a well-known necessary optimality condition as follows.

Theorem 3.1 *In the above notations, suppose that f is differentiable at $\overline{x}$. If $\overline{x}$ is a local solution of (P_S) then*

$$\nabla f(\overline{x})(u) \ge 0, \quad \forall u \in T_B(S, \overline{x}). \tag{4}$$

Of course, this result can be easily generalized to the nondifferentiable setting by replacing $\nabla f(\overline{x})$ by various generalized derivatives of f at $\overline{x}$. The converse of this result is not true and when we want to get a sufficient optimality condition then we need to impose a stronger condition than (4): fortunately, we get as well a stronger type of solution which we define below.

Definition 3.2 Let m be a positive integer and $\mu > 0$. One says that $\overline{x} \in S$ is a strict local solution of order m (and constant μ) for (P_S) if there exists a neighborhood U of $\overline{x}$ s.t. for every $x \in U \cap S$ one has

$$f(x) - f(\overline{x}) \ge \mu \|x - \overline{x}\|^m.$$

Note that this notion is widely studied in the literature: see [14] and the references therein. When $U = X$ one says that $\overline{x}$ is a strict global solution of order m. In this paper we deal only with the case $m = 1$ and we call a strict local solution of order 1 simply as strict local solution and similarly for the global solution. Observe that the concept of strict local solution (of order 1) is specific to the case where f is not differentiable at $\overline{x}$ or the restriction is active (that is $\overline{x} \in S \setminus \operatorname{int} S$): it is easy to see that if f is differentiable at $\overline{x} \in \operatorname{int} S$, then $\overline{x}$ cannot be a strict local solution.

We present now a basic sufficient condition for optimality. However, even if this is a known result, we give a proof for the reader's convenience.

Theorem 3.3 *Suppose that f is differentiable at $\overline{x} \in S$ and S is approximated at $\overline{x}$ by $T_B(S, \overline{x})$. If there exists $\mu > 0$ s.t.*

$$\nabla f(\overline{x})(u) \ge \mu \|u\|, \quad \forall u \in T_B(S, \overline{x}) \tag{5}$$

then $\overline{x}$ is a strict local solution of (P_S).

Proof Suppose, by way of contradiction, that for every positive integer n there exists $x_n \in S \cap B(\overline{x}, n^{-1})$ s.t.

$$f(x_n) < f(\overline{x}) + n^{-1} \|x_n - \overline{x}\|.$$

Since S is approximated at $\overline{x}$ by $T_B(S, \overline{x})$ there exists $\alpha : S \to T_B(S, \overline{x})$ s.t.

$$\lim_{x \xrightarrow{S} \overline{x}} \frac{\alpha(x) - (x - \overline{x})}{\|x - \overline{x}\|} = 0.$$

By the first-order Taylor expansion of f at $\overline{x}$ we get a sequence of real numbers $(\gamma_n) \to 0$ s.t. for every n

$$f(x_n) = f(\overline{x}) + \nabla f(\overline{x})(x_n - \overline{x}) + \gamma_n \|x_n - \overline{x}\|.$$

Therefore,

$$\begin{aligned} n^{-1} \|x_n - \overline{x}\| &> \nabla f(\overline{x})(x_n - \overline{x}) + \gamma_n \|x_n - \overline{x}\| \\ &= \nabla f(\overline{x})(\alpha(x_n)) + \nabla f(\overline{x})(x_n - \overline{x} - \alpha(x_n)) + \gamma_n \|x_n - \overline{x}\| \\ &\geq \mu \|\alpha(x_n)\| + \nabla f(\overline{x})(x_n - \overline{x} - \alpha(x_n)) + \gamma_n \|x_n - \overline{x}\| . \end{aligned}$$

Now, we divide by $\|x_n - \overline{x}\| > 0$ and we pass to the limit as $n \to \infty$ to get $0 \geq \mu$, i.e., a contradiction which ends the proof. □

Note that if X is finite dimensional or S is convex, the condition on S is automatically satisfied. Moreover, in the finite dimensional case (which was studied for the first time in [7]), the condition to exist $\mu > 0$ s.t.

$$\nabla f(\overline{x})(u) \geq \mu \|u\|, \quad \forall u \in T_B(S, \overline{x})$$

is equivalent to

$$\nabla f(\overline{x})(u) > 0, \quad \forall u \in T_B(S, \overline{x}) \backslash \{0\}$$

and the converse of Theorem 3.3 holds as well. Note that in this case, the above theorem and its converse was proved in a nondifferentiable setting in [14, Proposition 2.2].

A sufficient condition for strict efficiency in the convex not (necessarily) differentiable case is presented below. As we shall see, the optimality condition, even in a different form, is similar to that in Theorem 3.3.

Proposition 3.4 *Suppose that f is convex, S is convex and the following condition holds: there exists $\mu > 0$ s.t.*

$$\mu U_{X^*} \subset \partial f(\overline{x}) + N(S, \overline{x}). \tag{6}$$

Then $\overline{x}$ is a global strict solution of constant μ for (P_S).

Proof Let $x \in S$. Then $x - \overline{x} \in T(S, \overline{x})$. As above, there exists $x^* \in U_{X^*}$ s.t.

$$x^*(x - \overline{x}) = \|x - \overline{x}\|.$$

Following the assumption we made, there exists $u^* \in \partial f(\overline{x})$ s.t.

$$\mu x^* - u^* \in N(S, \overline{x})$$

whence

$$(\mu x^* - u^*)(x - \overline{x}) \leq 0.$$

This is

$$\mu x^*(x - \overline{x}) \leq u^*(x - \overline{x}) \leq f(x) - f(\overline{x}),$$

and, consequently,

$$\mu \|x - \overline{x}\| \leq f(x) - f(\overline{x}).$$

This ends the proof since x was arbitrarily chosen in S. □

For a similar result in literature, we can quote here [14, Theorem 2.2] where the setting is finite dimensional and a continuity condition on f and some generalized convexity assumptions on both f and S are used. In contrast, Proposition 3.4 holds for general Banach spaces and does not assume any continuity property of the objective function f.

Remark 3.5 We observe that the inequality in (5) can be replaced by an inclusion when one passes to the normals. This is based on the observation that if $u^* \in X^*, \overline{x} \in S \subset X$ and $\mu > 0$ then the condition

$$u^*(u) \geq \mu \|u\|, \quad \forall u \in T_B(S, \overline{x}) \tag{7}$$

is equivalent to

$$\mu U_{X^*} \subset u^* + N_B(S, \overline{x}). \tag{8}$$

Indeed, we show first that (7) implies (8). Take $x^* \in U_{X^*}$ and $u \in T_B(S, \overline{x})$. Then $(\mu x^* - u^*)(u) = \mu x^*(u) - u^*(u) \leq \mu x^*(u) - \mu \|u\| \leq 0$ for every $u \in T_B(S, \overline{x})$, whence $\mu x^* - u^* \in N_B(S, \overline{x})$. Conversely, take $u \in T_B(S, \overline{x})$. It is well known that $\|u\| = \max_{x^* \in U_{X^*}} x^*(u)$ so there exists $x^* \in U_{X^*}$ s.t. $\|u\| = x^*(u)$. On the other hand, $\mu x^* - u^* \in N_B(S, \overline{x})$, then $(\mu x^* - u^*)(u) \leq 0$ whence $\mu \|u\| \leq u^*(u)$.

In view of this remark, if f is differentiable at $\overline{x}$, then the relation (6) is equivalent to (5) since $\partial f(\overline{x}) = \{\nabla f(\overline{x})\}$, whence in that case Proposition 3.4 is implied by Theorem 3.3.

The next step is to consider the question if Proposition 3.4 remains true when one deals with nonconvex, nonsmooth functions and when one uses the symbols ∂ and N in the sense of Mordukhovich's generalized differentiation objects.

Of course, any necessary optimality condition for local strict minima should cover the known necessary conditions for the local minima as well. First, we remind an important result for local minima.

Theorem 3.6 [12, Proposition 5.3] *Suppose that X is an Asplund space and f is Lipschitz around $\overline{x}$. If $\overline{x}$ is a local solution for (P_S) then*

$$0 \in \partial f(\overline{x}) + N(S, \overline{x}).$$

We present now our result concerning the optimality conditions for strict efficiency. See also [13, Corollary 4.4].

Theorem 3.7 *Suppose that X is an Asplund space and f is Lipschitz around $\overline{x}$. If $\overline{x}$ is a strict local solution of constant $\mu > 0$ for (P_S) then*

$$\mu U_{X^*} \subset \partial f(\overline{x}) + N(S, \overline{x}).$$

Proof Since $\overline{x}$ is a strict local solution of constant μ for (P_S) there exists a neighborhood U of $\overline{x}$ s.t. for every $x \in U \cap S$,

$$f(x) - f(\overline{x}) \geq \mu \|x - \overline{x}\|.$$

This means that the function $\varphi : X \to \mathbb{R}, \varphi(x) = f(x) - f(\overline{x}) - \mu \|x - \overline{x}\|$ achieves at $\overline{x}$ a local minimum on S. By the use of infinite penalization technique, $\overline{x}$ is an unconstrained local minimum point for $\varphi + \delta_S$. Then,

$$0 \in \hat{\partial}\left(f(\cdot) - f(\overline{x}) - \mu \|\cdot - \overline{x}\| + \delta_S(\cdot)\right)(\overline{x}).$$

We denote $\varphi_1 : X \to \overline{\mathbb{R}}, \varphi_1(x) = f(x) - f(\overline{x}) + \delta_S(x)$ and $\varphi_2 : X \to \mathbb{R}, \varphi_2(x) = -\mu \|x - \overline{x}\|$.

Note that $-\varphi_2$ is a convex function and

$$\begin{aligned}\hat{\partial}^+\varphi_2(\overline{x}) &= -\hat{\partial}(-\varphi_2)(\overline{x})\\ &= -\partial(-\varphi_2)(\overline{x}) = \mu U_{X^*} \neq \emptyset.\end{aligned}$$

Then we can apply the calculus rule given in (3) to get

$$0 \in \bigcap_{x^* \in \mu U_{X^*}} [x^* + \hat{\partial}\varphi_1(\overline{x})].$$

Hence,

$$\begin{aligned}\mu U_{X^*} &\subset \hat{\partial}\varphi_1(\overline{x}) = \hat{\partial}\left(f(\cdot) - f(\overline{x}) + \delta_S(\cdot)\right)(\overline{x})\\ &\subset \partial\left(f(\cdot) - f(\overline{x}) + \delta_S(\cdot)\right)(\overline{x}).\end{aligned}$$

Since $f(\cdot) - f(\overline{x})$ is a Lipschitz function and δ_S is lower semicontinuous (note that S is closed) we can apply the exact sum rule, whence

$$\begin{aligned}\mu U_{X^*} &\subset \partial(f(\cdot) - f(\overline{x}))(\overline{x}) + \partial\delta_S(\overline{x})\\ &= \partial f(\overline{x}) + N(S, \overline{x}).\end{aligned}$$

For the last equality we have applied the obvious remark that $\partial(f(\cdot) - f(\overline{x}))(u) = \partial f(u)$ for every $u \in X$. The proof is complete. □

In the convex case (i.e., f convex and S convex) Theorem 3.7 is a partial converse of Proposition 3.4. Moreover, we record the following consequences.

Corollary 3.8 *Suppose that X is an Asplund space and f is continuously differentiable around $\overline{x}$. If $\overline{x}$ is a strict local solution of constant $\mu > 0$ for (P_S) then*

$$\mu U_{X^*} \subset \nabla f(\overline{x}) + N(S, \overline{x}).$$

Proof If f is continuously differentiable around $\overline{x}$ then it is Lipschitz around $\overline{x}$ and $\partial f(\overline{x}) = \{\nabla f(\overline{x})\}$ (see [12, Corollary 1.82]). □

Corollary 3.9 *Suppose that X is an Asplund and $S = X$ (the unconstrained case). If $\overline{x}$ is a strict local solution of constant $\mu > 0$ for (P_S) then*

$$\mu U_{X^*} \subset \hat{\partial} f(\overline{x}).$$

Proof Following the way of the proof of Theorem 3.7, in this case we don't need to penalize the initial function f, so we don't need to apply the sum rule for the basic subdifferential. Hence the Lipschitz condition on f can be dropped. □

We illustrate these results by some examples.

Example 3.10 Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} 0, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$ and $S = \mathbb{R}$. Then is easy to see that $\partial f(0) = \{-1\}$ whence, following Corollary 3.9, 0 is not a strict local solution for (P_S).

Example 3.11 Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. This function is Lipschitz and $\partial f(0) = [-1, 1]$ (see [12, p. 87]). However, it is easy to see that 0 is not a strict local solution for (P_S) (for $S = \mathbb{R}$) whence the converse of Theorem 3.7 does not hold.

We present now a partial converse of Theorem 3.7 which can be seen as well as a partial generalization of Proposition 3.4.

Proposition 3.12 *Let $\overline{x} \in S$, $\mu > 0$, $\nu \in (0, \mu)$. Suppose that S is convex, and there exist a finite number of Fréchet subgradients $u_1^*, u_2^*, \ldots, u_k^* \in \widehat{\partial} f(\overline{x})$ s.t.*

$$\mu U_{X^*} \subset \{u_1^*, u_2^*, \ldots, u_k^*\} + N(S, \overline{x}). \tag{9}$$

Then $\overline{x}$ is a local strict solution of constant ν for (P_S).

Proof We use the analytic characterization of the Fréchet subgradients: for every $i = 1, 2, \ldots, k$ there exists a function s_i differentiable at $\overline{x}$ s.t. $u_i^* = \nabla s_i(\overline{x})$ and $s_i(\overline{x}) = f(\overline{x})$, $s_i(x) \leq f(x)$ for every $x \in X$. Since s_i is differentiable, there exists $\alpha_i : X \to \mathbb{R}$, $\lim_{h \to 0} \alpha_i(h) = \alpha_i(0) = 0$ s.t. for every $x \in X$, one has

$$s_i(x) - s_i(\overline{x}) = \nabla s_i(\overline{x})(x - \overline{x}) + \|x - \overline{x}\| \alpha_i(x - \overline{x}).$$

Since every α_i is continuous at 0 and takes the value 0 at 0, there exists a neighborhood U of $\overline{x}$ s.t. for every $x \in U$, and every $i = 1, 2, \ldots, k$

$$|\alpha_i(x - \overline{x})| \leq \mu - \nu.$$

Take now $x \in S \cap U$. Then $x - \overline{x} \in T(S, \overline{x})$. As above, there exists $x^* \in U_{X^*}$ s.t.

$$x^*(x - \overline{x}) = \|x - \overline{x}\|.$$

By the hypothesis, there exists an index i s.t.

$$\mu x^* - u_i^* \in N(S, \overline{x})$$

whence

$$\left(\mu x^* - u_i^*\right)(x - \overline{x}) \leq 0.$$

Consequently,

$$\begin{aligned} \mu \|x - \overline{x}\| &\leq u_i^*(x - \overline{x}) \\ &= \nabla s_i(\overline{x})(x - \overline{x}) \\ &= s_i(x) - s_i(\overline{x}) - \|x - \overline{x}\| \alpha_i(x - \overline{x}) \\ &\leq f(x) - f(\overline{x}) - \|x - \overline{x}\| \alpha_i(x - \overline{x}). \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) - f(\overline{x}) &\geq (\mu + \alpha_i(x - \overline{x})) \|x - \overline{x}\| \\ &\geq (\mu - \mu + \nu) \|x - \overline{x}\| \\ &= \nu \|x - \overline{x}\|, \end{aligned}$$

and the proof is complete. □

Note that, if f is lower regular at $\overline{x}$, then one can take in this result basic subgradients instead of Fréchet subgradients.

4 Vectorial case

In this section we consider an optimization problem with vectorial cost function. Let X, Y be normed vector spaces. We consider a pointed closed convex proper cone $K \subset Y$ which introduces a partial order on Y by the equivalence $y_1 \leq_K y_2 \Leftrightarrow y_2 - y_1 \in K$. The set $K^* := \{y^* \in Y^* \mid y^*(y) \geq 0, \forall y \in Y\}$ is the dual cone of K. Let $f : X \to Y$ be a function and $S \subset X$ be a closed set. The problem to study is now

$$(P_V) \quad \min f(x), \quad x \in S.$$

For this problem, the sense of solution which corresponds to the classical concept in the scalar case is that of Pareto (local) solution with respect to K: one says that $\overline{x} \in S$ is a local Pareto solution for (P_V) if there exists a neighborhood U of $\overline{x}$ s.t. for every $x \in S \cap U \backslash \{\overline{x}\}$, $f(x) - f(\overline{x}) \notin -K$. If $\operatorname{int} K \neq \emptyset$ (in which case one says that K is a solid cone) then one can speak as well about weak solutions: one says that $\overline{x} \in S$ is a local weak Pareto solution for (P_V) if there exists a neighborhood U of $\overline{x}$ s.t. for every $x \in S \cap U$, $f(x) - f(\overline{x}) \notin -\operatorname{int} K$.

The sense of solution for (P_V) we shall mainly deal in the sequel is that of strict solution. In the vectorial case the notion of strict efficiency was introduced more than 20 years ago: we refer to [1,5,9] for comments and historical facts. In fact, there are at least two ways to define this notion, by means of two scalarization functions. We remind that if $y \in Y$ and $M \subset Y$ the oriented distance function introduced in [8] is defined as $\Delta(y, M) := \mathrm{d}(y, M) - \mathrm{d}(y, Y \setminus M)$ where, as usual, $\mathrm{d}(y, M) := \inf_{u \in M} \|y - u\|$.

Definition 4.1 Let m be a positive integer and $\mu > 0$. A point $\overline{x} \in S$ is called local strict solution of order m (and constant μ) of (P_V) if there exists a neighborhood U of $\overline{x}$ s.t. for every $x \in U \cap S$, $\mathrm{d}(f(x) - f(\overline{x}), -K) \geq \mu \|x - \overline{x}\|^m$.

Remark 4.2 This definition appears in [9,10]. Note that in [5] the authors have studied in detail a notion which, formally, is not far from the above defined concept: it requires $\Delta(f(x) - f(\overline{x}), -K) \geq \mu \|x - \overline{x}\|^m$ in the last part of the above definition (see also [1, Chapter 8]). Observe that in fact these concepts do coincide. One implication is clear since $\mathrm{d}(f(x) - f(\overline{x}), -K) \geq \Delta(f(x) - f(\overline{x}), -K)$. Suppose that $\overline{x} \in S$ satisfies Definition 4.1. If $x = \overline{x}$ we have nothing to prove. Otherwise, $\mathrm{d}(f(x) - f(\overline{x}), -K) > 0$ whence $f(x) - f(\overline{x}) \notin -K$, i.e., $f(x) - f(\overline{x}) \in Y \backslash -K$. This shows that $\mathrm{d}(f(x) - f(\overline{x}), Y \backslash -K) = 0$, whence $\mathrm{d}(f(x) - f(\overline{x}), -K) = \Delta(f(x) - f(\overline{x}), -K)$ and the thesis is proved.

As in the scalar case, we consider local strict solution of order 1 which we call, for simplicity, local strict solution. Observe that a local strict solution is a Pareto local solution and if $-\operatorname{int} K \neq \emptyset$, then it is local weak Pareto minimum as well. Moreover, observe that in the case $Y = \mathbb{R}$, $K = \mathbb{R}_+$ we get the scalar notion studied in the previous section.

In the above quoted works, the notion of local strict solution was studied (mainly) in the context of finite dimensional vector spaces and by means of various generalized derivatives which are based on the classical tangency concepts. In this paper we address a different view to the problem, namely we try to keep the setting of infinite dimensional spaces as long as possible, and to work with the normal or Fréchet subgradients. For optimality conditions using generalized derivatives we refer to [10].

In order to conclude this introductory part for the vectorial case, let us remind a subdifferential chain rule (see [12, Corollary 3.43]). Recall ([12, Definition 3.25]) that a function $f : X \to Y$ is said to be strictly Lipschitz at $\overline{x}$ if it is locally Lipschitzian around this point and there exists a neighborhood V of the origin in X s.t. the sequence $(t_k^{-1}(f(x_k + t_k v) - f(x_k)))_{k\in\mathbb{N}}$ contains a norm convergent subsequence whenever $v \in V$, $x_k \to \overline{x}$, $t_k \downarrow 0$.

- Suppose that X, Y are Asplund spaces. Let $f : X \to Y$ and $\varphi : Y \to \mathbb{R}$ s.t. f is strictly Lipschitz at $\overline{x} \in X$ and φ is Lipschitz around $f(\overline{x})$; then

$$\partial(\varphi \circ f)(\overline{x}) \subset \bigcup_{y^*\in\partial\varphi(f(\overline{x}))} \partial\left(y^* \circ f\right)(\overline{x}).$$

In the vectorial setting, having $-\operatorname{int} K \neq \emptyset$ is an important advantage, this condition being very important in a theoretical discussion. But, this is quite restrictive because for many important particular Banach spaces, the natural ordering cones have empty interiors. However, the case of cones with nonempty interior should be considered because it still corresponds to some infinite dimensional Banach spaces (as $C[0, 1]$ for example) and to the finite dimensional spaces as well. Taking into account these considerations, we shall later split this section in two subsections. We remind first a necessary optimality condition for weak minima.

Theorem 4.3 (see [3, Theorem 3.1]) *Suppose that X, Y are Asplund spaces and f is strictly Lipschitz. Suppose that $\overline{x} \in S$ is a weak solution for (P_V). There exists $v^* \in K^*\backslash\{0\}$ s.t.*

$$0 \in \partial\left(v^* \circ f\right)(\overline{x}) + N(S, \overline{x}).$$

Before considering the separate cases we have mentioned above, we want to point out why a necessary condition for strict minima which can be obtained directly from definition is not suitable. Observe that the Definition 4.1 ensures that $\overline{x} \in S$ is a local strict solution of constant μ for (P_V) if and only if it is a local strict minimum of the same constant for the scalar problem

$$\min d_{-K}(f(\cdot) - f(\overline{x})), \quad x \in S.$$

Therefore, under the assumptions that X, Y are Asplund spaces and f is strictly Lipschitz, we can apply Theorem 3.7 and the chain rule to write successively,

$$\begin{aligned} \mu U_{X^*} &\subset \partial d_{-K}(f(\cdot) - f(\overline{x}))(\overline{x}) + N(S, \overline{x}) \\ &\subset \bigcup_{y^*\in\partial d_{-K}(0)} \partial\left(y^* \circ f\right)(\overline{x}) + N(S, \overline{x}). \end{aligned} \tag{10}$$

This condition has two major deficiencies: it does not cover the scalar case and the necessary condition for weak minima given in Theorem 4.3. To be explicit, let us consider first the case $Y = \mathbb{R}$ and $K = \mathbb{R}_+$. In this case, $\partial d_{-K}(0) = [0, 1]$ and the condition (10) becomes

$$\mu U_{X^*} \subset [0, 1]\partial f(\overline{x}) + N(S, \overline{x})$$

which is clearly weaker than the condition in Theorem 3.7. In the general case, if $-\operatorname{int} K \neq \emptyset$, a strict solution is a weak solution as well, but condition (10) does not cover condition in Theorem 4.3 because $0 \in \partial d_{-K}(0)$, so one loses the essential condition $v^* \neq 0$.

4.1 The case of solid ordering cone

Throughout this subsection we consider that $-\operatorname{int} K \neq \emptyset$. Now introduce into discussion a well-known separating functional (see [6, Section 2.3]) which we use in the sequel as a main tool. The symbol ∂ denotes the Fenchel subdifferential of a convex function (see [3]).

Lemma 4.4 *Let $K \subset Y$ be a closed convex cone with nonempty interior and let $e \in \operatorname{int} K$. Define the functional $\varphi_e : Y \to \mathbb{R}$ as*

$$\varphi_e(y) = \inf\{\lambda \in \mathbb{R} \mid y \in \lambda e - K\}.$$

This map is continuous, convex, Lipschitz and for every $\lambda \in \mathbb{R}$

$$\{y \mid \varphi_e(y) \le \lambda\} = \lambda e - K, \quad \{y \mid \varphi_e(y) < \lambda\} = \lambda e - \operatorname{int} K.$$

Moreover, for every $u \in Y$,

$$\partial\varphi_e(u) = \left\{v^* \in K^* \mid v^*(e) = 1, v^*(u) = \varphi_e(u)\right\}.$$

In the next lemma we characterize the local strict solution by means of the functional φ_e and this will allow us to look at this notion in a different way. This lemma is essentially proved in [2] for the set-valued case so we present now its (shorter) proof in our specific case.

Lemma 4.5 *Suppose that* $\operatorname{int} K \ne \emptyset$. *The point $\overline{x} \in S$ is a local strict solution of order m of (P_V) if and only if there exists a neighborhood U of $\overline{x}$ s.t. for every $e \in \operatorname{int} K$ there exists $\nu > 0$ s.t. for every $x \in U \cap S$, $\varphi_e(f(x) - f(\overline{x})) \ge \nu \|x - \overline{x}\|^m$.*

Proof Suppose that $\overline{x} \in S$ is a local strict solution of order m of (P_V) in the sense of Definition 4.1. Take U from that definition. Take $e \in \operatorname{int} K$ with $\|e\| = 1$. If $x = \overline{x}$ then obviously $\varphi_e(f(x) - f(\overline{x})) = 0$, whence the inequality in the conclusion trivially holds for every $\nu > 0$. Suppose that $x \ne \overline{x}$ is a point in $U \cap S$. Since $\mathrm{d}\left(f(x) - f(\overline{x}), -K\right) \ge \mu \|x - \overline{x}\|^m$ we obtain that

$$f(x) - f(\overline{x}) \notin \nu \|x - \overline{x}\|^m e - \operatorname{int} K$$

for any $\nu \in (0, \mu)$ (because otherwise we would obviously have $\mathrm{d}\left(f(x) - f(\overline{x}), -K\right) \le \nu \|x - \overline{x}\|^m < \mu \|x - \overline{x}\|^m$). Therefore, $\varphi_e(f(x) - f(\overline{x})) \ge \nu \|x - \overline{x}\|^m$ for any $\nu \in (0, \mu)$ and the conclusion is true even for $\nu = \mu$ (one passes to the limit with $\nu \to \mu$) for the element e we have fixed before. In order to show that it is valid for any other $e' \in \operatorname{int} K$ (eventually with another constant ν) observe that there exists a positive number a s.t. $ae - e' \in K$ and this implies that $\varphi_{e'}(u) \ge a^{-1}\varphi_e(u)$.

We prove now the converse implication. Let U be a neighborhood of $\overline{x}$, $e \in \operatorname{int} K$ and $\nu > 0$ s.t. for every $x \in U \cap S$, $\varphi_e(f(x) - f(\overline{x})) \ge \nu \|x - \overline{x}\|^m$. Fix $x \in U \cap S$. If $x = \overline{x}$ then the inequality $\mathrm{d}(f(x) - f(\overline{x}), -K) \ge \mu \|x - \overline{x}\|^m$ is clearly satisfied for any μ. Suppose that $x \ne \overline{x}$. Since $e \in \operatorname{int} K$ there exists $\rho > 0$ s.t. $e - D(0, \rho) \subset \operatorname{int} K$, so

$$D\left(0, \nu\rho \|x - \overline{x}\|^m\right) \subset \nu \|x - \overline{x}\|^m e - \operatorname{int} K.$$

We claim that

$$\mathrm{d}(f(x) - f(\overline{x}), -K) \ge \nu\rho \|x - \overline{x}\|^m.$$

Indeed, in the contrary case, there would exist $k \in K$ s.t.

$$\|f(x) - f(\overline{x}) + k\| < \nu\rho \|x - \overline{x}\|^m,$$

whence

$$\begin{aligned} f(x) - f(\overline{x}) &\in -k + D(0, \nu\rho \|x - \overline{x}\|^m) \\ &\subset -k + \nu \|x - \overline{x}\|^m e - \operatorname{int} K \\ &\subset \nu \|x - \overline{x}\|^m e - \operatorname{int} K, \end{aligned}$$

so

$$\varphi_e(f(x) - f(\overline{x})) < \nu \|x - \overline{x}\|^m$$

in contradiction with our hypothesis. Consequently, the claim is proved and $\mathrm{d}(f(x) - f(\overline{x}), -K) \geq \nu\rho \|x - \overline{x}\|^m$. Further, observe that $\mu := \nu\rho$ does not depend on x, and this allows us to conclude the proof. □

Now we are interested in the counterparts of the results from the scalar case.

Theorem 4.6 *Suppose that X, Y are Asplund spaces and f is strictly Lipschitz. Let $e \in \operatorname{int} K \cap S_Y$. If $\overline{x} \in S$ is a local strict solution of constant μ for (P_V) then*

$$\mu U_{X^*} \subset \bigcup_{y^* \in \partial\varphi_e(0)} \partial \left(y^* \circ f\right)(\overline{x}) + N(S, \overline{x}). \tag{11}$$

Proof We have seen in the proof of Lemma 4.5 that if $e \in \operatorname{int} K \cap S_Y$ and $\overline{x} \in S$ is a local strict solution of constant μ for (P_V) then it is local strict minimum of the same constant for

$$\min \varphi_e(f(\cdot) - f(\overline{x})), \quad x \in S.$$

We apply Theorem 3.7 and we get (note that φ_e is Lipschitz)

$$\begin{aligned} \mu U_{X^*} &\subset \partial\varphi_e(f(\cdot) - f(\overline{x}))(\overline{x}) + N(S, \overline{x}) \\ &\subset \bigcup_{y^* \in \partial\varphi_e(0)} \partial \left(y^* \circ f\right)(\overline{x}) + N(S, \overline{x}), \end{aligned}$$

and that is the conclusion. □

Let us observe that relation (11) solves both deficiencies we have noticed in relation (10). Indeed, if $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, then $\partial\varphi_e(0) = \{1\}$ and we get the conclusion of the scalar case. Moreover, in the general case, $0 \notin \partial\varphi_e(0)$ (see Lemma 4.4). In particular, these remarks show, on the basis of Example 3.11, that the converse of Theorem 4.6 does not hold.

Example 4.7 Let $X := \mathbb{R}^2$, $Y := \mathbb{R}^2$, $K := \mathbb{R}^2_+$. Note that for $e = (1, 1) \in \operatorname{int} K$, $\partial\varphi_e(0, 0) = \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 + x_2 = 1\}$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (||x_1| + x_2|, ||x_1| + x_2|)$. This function is Lipschitz on $\mathbb{R}^2$, whence strictly Lipschitz because Y has finite dimension. Then

$$\bigcup_{y^* \in \partial\varphi_e(0,0)} \partial \left(y^* \circ f\right)(0, 0) = \{(x_1, x_2) \mid |x_1| \leq x_2 \leq 1\} \cup \{(x_1, x_2) \mid x_2 = -|x_1|, -1 \leq x_1 \leq 1\}$$

(see [12, pp. 244, 245]). Obviously, if one takes $S := \mathbb{R}^2$, relation (11) does not hold for any $\mu > 0$, so $(0, 0)$ is not a local strict solution for (P_V). Note that $(0, 0)$ is a local weak Pareto solution for (P_V).

However, a partial converse of Theorem 4.6 is valid under convexity assumptions. One says that $f : X \to Y$ is K-convex if for every $x_1, x_2 \in X$ and every $\alpha \in (0, 1)$,

$$\alpha f(x_1) + (1 - \alpha) f(x_2) \in f(\alpha x_1 + (1 - \alpha)x_2) + K.$$

Theorem 4.8 *Suppose that f is K-convex, S is convex. Take $e \in \operatorname{int} K \cap S_Y, \overline{x} \in S$. If relation (11) holds, then $\overline{x}$ is a (global) strict solution for (P_V).*

Proof Observe (see Lemma 4.4) that $\partial\varphi_e(0) \subset K^*$ whence $y^* \circ f$ is convex for any $y^* \in \partial\varphi_e(0)$. The rest of the argument repeats the steps in the proof of Proposition 3.4. Take $x \in S$; there exists $x^* \in U_{X^*}$ s.t. $\mu x^*(x - \overline{x}) = \mu \|x - \overline{x}\|$. By hypothesis, there exists $y^* \in \partial\varphi_e(0)$ and $u^* \in \partial(y^* \circ f)(\overline{x})$ with

$$\left(\mu x^* - u^*\right)(x - \overline{x}) \leq 0.$$

This yields

$$\begin{aligned}\mu \|x - \overline{x}\| &\leq u^*(x - \overline{x}) \\ &\leq \left(y^* \circ f\right)(x) - \left(y^* \circ f\right)(\overline{x}) \\ &= y^*(f(x) - f(\overline{x})) \\ &\leq \varphi_e(f(x) - f(\overline{x})),\end{aligned}$$

and following Lemma 4.5 this ensures that $\overline{x}$ is a strict solution for (P_V). □

4.2 The case of non-solid ordering cone

Suppose that K has empty interior. In this context, in order to avoid the problems raised by formula (10), we take a different approach based on a modification of the solution concept. Take $e \in K$. We say that $\overline{x} \in S$ is a local e-strict solution of constant $\mu > 0$ for (P_V) if there exists a neighborhood U of $\overline{x}$ s.t. for every $x \in S \cap U$:

$$d_{-K-e}\left(f(x) - f(\overline{x})\right) - \mu \|x - \overline{x}\| \geq d_{-K-e}(0). \tag{12}$$

The idea to perturb the cone with an element in K is used in [4]. Since $0 \notin -K - e$, the above relation implies $f(x) - f(\overline{x}) \notin -K - e$ so, in particular, a local e-strict solution is an approximate solution in sense of [4].

If $d_{-K-e}(0) = \|e\|$, taking into account that $d_{-K-e}(y) \leq d_{-K}(y) + \|e\|$ for any $y \in Y$, relation (12) implies that $\overline{x}$ is a local strict solution for (P_V). Note also that the equality $d_{-K-e}(0) = \|e\|$ holds provided that the norm is K-monotone, i.e., for any $y_1, y_2 \in K$ with $y_2 - y_1 \in K$, one has $\|y_2\| \geq \|y_1\|$.

Note that the advantage of considering d_{-K-e} instead of d_{-K} consists of the fact that $\partial d_{-K-e}(0) \subset K^* \backslash \{0\}$ (see [4, Remark 2.2]). Then we get the result below.

Theorem 4.9 *Suppose that X, Y are Asplund spaces and f is strictly Lipschitz. Let $e \in K$. If $\overline{x} \in S$ is a local e-strict solution of constant μ for (P_V) then*

$$\mu U_{X^*} \subset \bigcup_{y^* \in \partial d_{-K-e}(0)} \partial\left(y^* \circ f\right)(\overline{x}) + N(S, \overline{x}). \tag{13}$$

Proof A local e-strict solution of constant μ for (P_V) is a local strict solution of constant μ for

$$\min d_{-K-e}\left(f(\cdot) - f(\overline{x})\right), \quad x \in S.$$

We apply Theorem 3.7 and we get (d_{-K-e} is Lipschitz)

$$\begin{aligned}\mu U_{X^*} &\subset \partial d_{-K-e}(f(\cdot) - f(\overline{x}))(\overline{x}) + N(S, \overline{x}) \\ &\subset \bigcup_{y^* \in \partial d_{-K-e}(0)} \partial\left(y^* \circ f\right)(\overline{x}) + N(S, \overline{x}),\end{aligned}$$

and that is the conclusion. □

Again, one can easily observe that in the scalar case, $\partial d_{-\mathbb{R}_+-1}(0) = \{1\}$. In the convex case one can prove a result similar to Theorem 4.8.

Theorem 4.10 *Suppose that f is K-convex and S is convex. Take $e \in S_Y$, $\overline{x} \in S$. If relation (13) holds, then $\overline{x}$ is a (global) e-strict solution for (P_V).*

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